L'Hopital Rule and limits at infinity

A useful technique in finding limits as $x \to c$ whether c is a constant or infinity!

Suppose $\lim_{x \to c} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} \\ \frac{\infty}{\infty} \\ 0 \cdot \infty \\ \infty - \infty \end{cases}$ "indeterminate cases"

Then $\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}} \neq \lim_{x \to c} \left(\frac{f(x)}{g(x)}\right)'$

Eg:

$$\lim_{x \to 1} \left[\frac{(x-1)}{x^2-1}\right] \longrightarrow \frac{0}{0} \longrightarrow \frac{x-1}{(x-1)(x+1)} \longrightarrow \frac{1}{x+1}$$

(Factoring technique)

$$= \lim_{x \to 1} \left(\frac{1}{x+1}\right) = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

New technique: L'Hopital rule

$$\lim_{x \to 1} \left[\frac{x-1}{x^2-1}\right] = \lim_{x \to 1} \frac{(x-1)'}{(x^2-1)'} = \lim_{x \to 1} \frac{1}{2x}$$

$$= \frac{1}{2(1)} = \boxed{\frac{1}{2}}$$ confirms the previous result!

Eg: $\lim\limits_{x\to\frac{\pi}{2}} \left(\dfrac{\cos^2 x}{1-\sin x}\right) \xrightarrow[\text{Factoring}]{\frac{0}{0}}$: $\lim\limits_{x\to\frac{\pi}{2}} \left(\dfrac{1-\sin^2 x}{1-\sin x}\right)$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)(1+\sin x)}{(1-\sin x)} = \lim_{x\to\frac{\pi}{2}} (1+\sin x)$$

$$= 1+\sin\left(\frac{\pi}{2}\right) = 1+1=2$$

• L'H. R

$$\lim_{x\to\frac{\pi}{2}} \frac{(\cos^2 x)'}{(1-\sin x)'} = \lim_{x\to\frac{\pi}{2}} \left(\frac{2\cos x(-\sin x)}{-\cos x}\right) \qquad (f^n)' = n f^{n-1}\cdot f'$$

$$= \lim_{x\to\frac{\pi}{2}} (2\sin x) = 2\sin\left(\frac{\pi}{2}\right) = 2(1) = 2$$

Ex: $\lim\limits_{x\to 0^+} (x\ln x) \xrightarrow{\text{case}} 0(-\infty)$

Solution: $\lim\limits_{x\to 0^+} \left(\dfrac{\ln x}{\frac{1}{x}}\right)$ $\quad \dfrac{f}{g} \longrightarrow \dfrac{\infty}{\infty}$ $\quad$ (Recall $\dfrac{k}{0} \longrightarrow \infty$)

$$\longrightarrow \lim_{x\to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) \qquad (\ln u)' = \frac{u'}{u}, \quad \frac{1}{u} = u^{-1}$$

$$\longrightarrow \lim_{x\to 0^+} \left(\frac{1}{x}\left(\frac{-x^2}{1}\right)\right) = \lim_{x\to 0^+} (-x) = 0$$

So $\boxed{\lim\limits_{x\to 0^+} (x\ln x) = 0}$

Ex: $\lim\limits_{x \to 0} \left(\dfrac{e^x - x - 1}{\cos x - 1} \right) \longrightarrow \dfrac{e^0 - 0 - 1}{\cos(0) - 1} = \dfrac{0}{0}$

$\lim\limits_{x \to 0} \left(\dfrac{e^x - 1}{-\sin x} \right) \longrightarrow \dfrac{e^0 - 1}{\sin(0)} = \dfrac{0}{0}$ repeat the L'hopital rule

$\lim\limits_{x \to 0} \left(\dfrac{e^x}{-\cos x} \right) = \dfrac{e^0}{-\cos(0)} = \dfrac{1}{-1} = \boxed{-1}$

Ex: $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x} \right) \longrightarrow$ "$\infty - \infty$" "$\dfrac{1}{0} - \dfrac{1}{0}$"

$= \lim\limits_{x \to 0} \left(\dfrac{x - \sin x}{x \sin x} \right) \longrightarrow$ "$\dfrac{0}{0}$" L'H.R

$= \lim\limits_{x \to 0} \left(\dfrac{1 - \cos x}{1 \cdot \sin x + x \cos x} \right) \longrightarrow$ "$\dfrac{0}{0}$" L'H.R

$= \lim\limits_{x \to 0} \left(\dfrac{+\sin x}{\cos x + 1 \cdot \cos x - x \sin x} \right) = \dfrac{0}{1 + 1 - 0} = \boxed{0}$

$$\lim_{x\to\infty}\left(\frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n}\right)$$

Ratio of two polynomials = Rational function

$$= \begin{cases} \frac{a_0}{b_0} & \text{if } m = n \quad (\text{same degree}) \\ 0 & \text{if } m < n \quad (\text{greater degree at the bottom}) \\ \infty & \text{if } m > n \quad (\text{greater degree at the top}) \end{cases}$$

Eg: $\lim_{x\to\infty}\left(\frac{3x^2-1}{x^1}\right)$ greater degree at top $\to \boxed{\infty}$

$\to \frac{\infty}{\infty} \to$ L'H $\to \lim_{x\to\infty}\left(\frac{6x}{1}\right) = \lim_{x\to\infty}(6x) = \infty$

Eg: $\lim_{x\to\infty}\left(\frac{1-x}{3x^2-2x}\right)$ greater degree at bottom $\to \boxed{0}$

$\frac{\infty}{\infty}$ L'HR $\to \lim_{x\to\infty}\left(\frac{-1}{6x-2}\right) \to \frac{k}{\infty} \to \boxed{0}$

Note: $\boxed{\frac{k}{\infty} \to 0}$

Eg: $\lim_{x\to\infty}\left(\frac{-x^2}{3+2x^2}\right)$ degree top = degree bottom $\to \boxed{\frac{-1}{2}}$

L'HR $\to \lim_{x\to\infty}\left(\frac{-2x}{4x}\right) \xrightarrow{\text{L'HR}} \lim_{x\to\infty}\left(\frac{-2}{4}\right) = \boxed{-\frac{1}{2}}$

Remark: if $\lim_{x\to\infty} f(x) = c$, $y = c$ is called a horizontal asymptote

example

Remark: Finding the limit of $(f(x))^{g(x)}$

STEP: say $\lim_{x\to c}\left[f(x)^{g(x)}\right]$. Let

$y = f(x)^{g(x)}$, take ln of both sides

to produce: $\ln y = g(x)\ln(f(x))$

If $\lim_{x\to c}(\ln y) = \lim_{x\to c}\left[g(x)\ln(f(x))\right] = k$

then $\boxed{\lim_{x\to c} y = \lim_{x\to c} f(x)^{g(x)} = e^{k}}$

Eg: $\lim_{x\to 0^+} x^{x} \longrightarrow$ "0^0", "1^∞", "∞^0",

let $y = x^x \longrightarrow \ln y = x\ln x$

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}(x\ln x) = \lim_{x\to 0^+}\left(\frac{\ln x}{\frac{1}{x}}\right) \xrightarrow{\text{L'HR}}$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim_{x\to 0^+}\left(-\frac{x}{1}\right) = 0$$

So $\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^x = e^0 = \boxed{1}$

Eg: $\lim_{x\to 0} (1+4x)^{\frac{1}{2x}}$

$y = (1+4x)^{\frac{1}{2x}} \longrightarrow \ln y = \frac{1}{2x}\ln(1+4x)$

$= \lim_{x\to 0}\left[\frac{1}{2x}\ln(1+4x)\right] \longrightarrow \frac{\ln(1)}{0} \longrightarrow \frac{0}{0}$

$= \lim_{x\to 0}\left[\frac{\frac{4}{1+4x}}{\frac{2}{1}}\right] = \lim_{x\to 0}\left[\frac{2}{1+4x}\right] \longrightarrow \frac{2}{1} = 2$

So, $\lim_{x\to 0}(1+4x)^{\frac{1}{2x}} = \boxed{e^2}$

Ex: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \longrightarrow \text{"}1^\infty\text{"}$

$y = \left(1+\frac{1}{x}\right)^x \longrightarrow \ln y = x\ln\left(1+\frac{1}{x}\right)$

$\lim_{x\to\infty}\left(x\ln\left(1+\frac{1}{x}\right)\right) \xrightarrow{\infty\cdot 0} \xrightarrow{\text{L'HR}} = \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right]$

$= \lim_{x\to\infty}\left[\frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{1}{1+\frac{1}{x}}\right] = \frac{1}{1+0} = 1$

So $\boxed{\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right] = e}$